MATH-2305 Discrete Mathematics

PREDICATES AND QUANTIFIERS
Truth Verification of those Statements that are not Propositions

• There are no rules in propositional logic that allow to conduct the truth value of the following statements:
  • “There is at least one student in this classroom who took three Calculus classes”
  • “x+7=10”
  • “Each computer in this classroom is connected to the network”
  • “y is a student”
Predicates

• Statements like
  ➢ \( x + 7 = 10 \)
  ➢ \( y \) is a student
  ➢ \( x \geq 3 \)
have two parts.
• The first part, a **variable**, is the subject of the statement
• The second part is the **predicate**. It refers to a property that the subject of the statement may have
Predicates

• **Predicate** is a dual-purpose term

• On the one hand, a **predicate** is a property or description of subjects. The following predicates are all shown in red and bold:
  – James **is tall**.
  – The bridge **is long**.
  – 19 **is a prime number**.

• On the other hand, a **predicate** is also used as a synonym of a **propositional function**, where the description is related not to a certain subject, but to a **variable**.
Predicates and Propositional Functions

• If a statement depends on a variable, this statement is called a propositional function (often a propositional function itself is called a predicate)

• We can denote such a statement by $P(x)$. 

• Once a value has been assigned to the variable $x$, the statement $P(x)$ becomes a proposition, which has a truth value.

• $P(x)$ is also referred to as a 1-place predicate or simply a predicate
Multivariable Propositional Functions

• Multivariable propositional functions depend on more than one variable. For example,
  ▪ $x$ is taller than $y$
  ▪ $a$ is greater than one of $b, c$
  ▪ $x$ is at least $n$ inches taller than $y$
  ▪ $x+y=z$
  ▪ $x$ and $y$ are students
$n$-ary Predicates

• A multivariable propositional function depends on more than one variable. For example, the propositional function $x+y=z$ depends on 3 variables.

• Once some values have been assigned to the variables $x$, $y$, $z$, the statement $x+y=z$ becomes a proposition, which has a truth value.

• A statement of the form $P(x_1, x_2, ..., x_n)$ is called an $n$-place predicate or $n$-ary predicate.
Propositional Functions and Programming

• Propositional functions are widely used in computer programming and algorithm design for branching programs and algorithms.

If \(<\text{propositional function}(x, y, z)\> \) then \(<\text{statement}\>\) else \(<\text{statement}\>\)

➢ For example:

If \(((x>0) \text{ and } (x<y)) \text{ or } (y>z))\) then \(<\text{statement}\>\) else \(<\text{statement}\>\)
Precondition and Postcondition

• **Precondition** is a statement that describes a valid input of some algorithm or program segment

• **Postcondition** is a statement that the output of some algorithm or program segment should satisfy when the algorithm (the program segment) has run

• Preconditions and Postconditions are widely used to verify the correctness of algorithms and programs
Precondition and Postcondition

• **Example.** Let we need to swap the values of two variables \(x\) and \(y\).

• **Algorithm:** \(\text{temp} := x \ ; \ x := y; \ y := \text{temp}\)

• **Precondition:** “\((x=a) \ & \ (y=b)\)"

• **Postcondition:** “\((x=b) \ & \ (y=a)\)"

• := means “set to” \(\rightarrow \ x := y\) means “set \(x\) equal to \(y\)”
Semantics

• If logical propositions are to have meaning, they need to describe something. Up to now, propositions and propositional functions such as “Johnny is tall”, “Debbie is 5 years old”, and “$x^2=-1$” had no intrinsic meaning. Either they are true or false, but no more.

• In order to make such propositions and propositional functions meaningful, we need to have a domain (or universe) of discourse, (or simply domain (or universe)) i.e. a collection of subjects about which the propositions relate.

• Question: What are the domains for the three propositions above?
Semantics

• **Question:** What are the domains for the following propositions and propositional functions?

  • “Johnny is tall”, “Debbie is 5 years old”
  
    **Possible answers:** {Johnny, Debbie}; {People in the world}; {People leaving in Texarkana}, {Students in this classroom}, etc.

  • “\(x^2=-1\)”

    **Possible answers:** C – the set of complex numbers; R – the set of real numbers, Z – the set of integer numbers

  • **Depending on the particular domain, a proposition may be true or false**
Semantics

• Semantics is very important in computer programming
• Each data structure and even each single variable that are used in a computer program always have their domain of discourse (type).
• For example, a variable \( x \) can be declared as real or integer or byte or char.
• Depending on this declaration, such expressions as \( x=3.5, x=5, x=‘Class’, x=-1 \) can be meaningful and acceptable or not.
Quantifiers

• There are statements, which assert that some property is true for all values of a variable in a particular domain ("Each student in this classroom takes the Discrete Math course in Fall semester 2014").

• There are also statements, which assert that there is an element in a particular domain with a certain property ("There is at least one person in this classroom who is not a student").

• Such statements can be formulated using quantifiers
Quantifiers

There are two quantifiers

• Universal Quantifier
  “∀” reads “for all” or “Each” or “Every”

• Existential Quantifier
  “∃” reads “there exists” or “there is at least one”

A quantifier is placed in front of a propositional function and binds it to obtain a proposition with a semantic value.
Quantifiers

• A statement (propositional function) $P(x)$ is quantified if there is a quantifier in front of it, which is applied to it:

$$\forall x \; P(X) \quad \exists x \; P(X)$$

- $\forall x \; P(x)$ is **TRUE** if $P(x)$ is true for every single $x$.
- $\forall x \; P(x)$ is **FALSE** if there is an $x$ for which $P(x)$ is false.

- $\exists x \; P(x)$ is **TRUE** if there is an $x$ for which $P(x)$ is true.
- $\exists x \; P(x)$ is **FALSE** if $P(x)$ is false for every single $x$. 
Quantifiers

- The truth table for quantifiers

<table>
<thead>
<tr>
<th>Statement</th>
<th>When True?</th>
<th>When False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \ P(X)$</td>
<td>P(x) is true for every x</td>
<td>There is if only one value of x for which P(x) is false</td>
</tr>
<tr>
<td>$\exists x \ P(X)$</td>
<td>There is an x for which P(x) is true</td>
<td>P(x) is false for every x</td>
</tr>
</tbody>
</table>
Universal Quantifier

• “∀x P(x)” is true when every instance of x makes P(x) true when plugged in
• Like conjunctioning over entire domain of P(x)

\[ ∀x P(x) \iff P(x_1) \land P(x_2) \land P(x_3) \land \ldots \]

• Example: Each non-negative real number has a square root
Existential Quantifier

• “$\exists x \ P(x)$” is true when an instance can be found which when plugged in for $x$ makes $P(x)$ true
• Like disjunctioning over entire domain of $P(x)$
  \[
  \exists x \ P(x) \iff P(x_1) \lor P(x_2) \lor P(x_3) \lor \ldots
  \]
• **Example**: There exist a complex number whose square is a negative real number
• $(i^2=-1, \ i \ is \ an \ imaginary \ unity)$
Multivariate Quantification

• Quantification involving only one variable is fairly straightforward. Just a bunch of OR’s or a bunch of AND’s.

• When two or more variables are involved each of which is bound by a quantifier, the order of the binding is important and the meaning often requires some thought.
Parsing Example

Question: If the domain for $x$, $y$, and $z$ is the natural numbers $\{0, 1, 2, 3, 4, 5, 6, 7, \ldots \}$ what’s the truth value of $\exists x \forall y \exists z \ P(x, y, z)$; $P(x, y, z) =$ “$y - x \geq z$”?

Answer: True.

• For any “exists” we need to find a positive instance.
• Since $x$ is the first variable in the expression and is “existential”, we need a number that works for all other $y, z$. Set $x = 0$ (want to ensure that $y - x$ is not too small).
• Now for each $y$ we need to find a positive instance $z$ such that $y - x \geq z$ holds. Plugging in $x = 0$ we need to satisfy $y \geq z$, so set $z = y$. 

Question: If the domain for $x$, $y$, and $z$ is the natural numbers $\{0,1,2,3,4,5,6,7,\ldots\}$, did we have to set $z = y$ to ensure that

$$\exists x \forall y \exists z \ P(x,y,z)$$

where $P(x,y,z) = "y - x \geq z"$ is true?

Answer: No. Could also have used the constant $z = 0$. There are other valid solutions.
Question: Isn’t it simpler to satisfy $\exists x \ \forall y \ \exists z \ (y - x \geq z)$ by setting $x = y$ and $z = 0$?

Answer: No, this is illegal! The existence of $x$ comes before we know about $y$. I.e., the scope of $x$ is higher than the scope of $y$. So we have to find first such $x$ that does not depend on $y$. Thus, it is illegal to choose $x$ depending on $y$. 
Order matters

• Set the domain of discourse to be all real numbers.
• Let $P(x,y) = "x < y"$.
• Question: What does $\forall x \exists y P(x,y)$ mean?
• Answer: $\forall x \exists y P(x,y)$:
  • “All numbers $x$ admit a larger number $y$”
• Question: What’s the truth value of this expression?
• Answer: True. For any real number there is a larger real number
Order matters

$P(x, y) = "x < y"$

• **Question:** What does $\exists y \forall x P(x, y)$ mean?

• **Answer:** $\exists y \forall x P(x, y)$:
  
  “Some number $y$ is larger than all $x$”

• **Question:** What’s the truth value of this expression?

• **Answer:** False. There is no the largest real number. Additionally, the number cannot be larger than itself.
Order matters – but not always

• If we have two quantifiers of the same kind, order does not matter.

• $\forall x \forall y$ is the same as $\forall y \forall x$ because these are both interpreted as “for every combination of $x$ and $y$…”

• $\exists x \exists y$ is the same as $\exists y \exists x$ because these are both interpreted as “there is a pair $x, y$…”
Predicates - the meaning of multiple quantifiers

- \( \forall x \forall y \ P(x,y) \): P(x,y) true for all (x, y) pairs.
- \( \exists x \exists y \ P(x,y) \): P(x,y) true for at least one (x, y) pair.
- \( \forall x \exists y \ P(x,y) \): For every value of x we can find a (possibly different) y so that P(x,y) is true.
- \( \exists x \forall y \ P(x,y) \): There is at least one x for which P(x,y) is always true.

Suppose P(x,y) = “x’s favorite class is y.”

Quantification order is not commutative in general!
Negation of Logical Expressions with Quantifiers

• What about $\forall x P(x)$ and $\exists x P(x)$?

• Since the quantifiers are the same as taking a bunch of AND’s ($\forall$) or OR’s ($\exists$) we obtain applying De Morgan’s laws:

$\forall x P(x) \equiv (P(x_1) \land P(x_2) \land \ldots \land P(x_n)) \equiv P(x_1) \lor P(x_2) \lor \ldots \lor P(x_n) \equiv \exists x \ P(x)$;

$\exists x P(x) \equiv (P(x_1) \lor P(x_2) \lor \ldots \lor P(x_n)) \equiv P(x_1) \land P(x_2) \land \ldots \land P(x_n) \equiv \forall x \ P(x)$.
Negation of Logical Expressions with Quantifiers

\[ \forall x \ P(x) \equiv \exists x \ P(x) \quad \exists x \ P(x) \equiv \forall x \ P(x) \]

- Negation of the expression containing the universal quantifier like “for every x belonging to some certain domain P(x) holds” means “there exist at least one x in the certain domain such that P(x) does not hold”

- Negation of the expression containing the existential quantifier like “there exist such x in the certain domain that P(x) holds” means “for every x from the certain domain P(x) does not hold”
Negation of Logical Expressions with Quantifiers

• General rule: to negate a logical expression containing quantifier (quantifiers), move negation to the expression under quantifier (quantifiers) and flip all quantifiers from $\forall$ to $\exists$ and vice versa.
Negation Example

Compute:

\[ \forall x \exists y \left( x^2 \leq y \right) \]

• In English, we are trying to find the opposite of “every \( x \) admits a \( y \) greater or equal to \( x \) squared”. The opposite is that “some \( x \) does not admit \( y \) greater or equal to \( x \) squared”

• Algebraically, one just flips all quantifiers from \( \forall \) to \( \exists \) and vice versa, and negates the interior propositional function. In our case we get:

\[ \forall x \exists y \left( x^2 \leq y \right) \equiv \exists x \forall y \left( x^2 \leq y \right) \equiv \exists x \forall y \left( x^2 > y \right) \]
Logical Equivalences Involving Predicates and Quantifiers

• Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain is used for the variables in these propositional functions.